CREATING LIMIT FUNCTIONS BY THE PANG-ZALCMAN LEMMA

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Abstract. In this paper we calculate the collection of limit functions obtained by applying an extension of Zalcman's Lemma, due to X.C Pang to the non-normal family $\{f(nz) : n \in \mathbb{N}\}$ in $\mathbb{C}$, where $f = Re^P$. Here R and P are an arbitrary rational function and a polynomial, respectively, where P is a non-constant polnomial.

Contents

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1. Introduction

A well-known powerful tool in the theory of normal families is the following lemma of L. Zalcman.

Zalcman's Lemma. [12] *A family $\mathcal{F}$ of functions meromorphic (resp., analytic) on the unit disk Δ is not normal if and only if there exist*
(a) a number $0 < r < 1$;
(b) points z_n, $|z_n| < r$;
(c) functions $f_n \in \mathcal{F}$; and
(d) numbers $\rho_n \to 0^+$,
such that

$$f_n(z_n + \rho_n\zeta) \overset{\chi}{\Rightarrow} g(\zeta) \quad (f_n(z_n + \rho_n\zeta) \Rightarrow g(\zeta)) \ ,$$

where g is a nonconstant meromorphic (entire) function on $\mathbb{C}$.

Morever, g can be taken to satisfy the normalization
$g^{\#}(\zeta) \leq g^{\#}(0) = 1$, $\zeta \in \mathbb{C}$.

Here and throughout the paper, '$\overset{\chi}{\Rightarrow}$' ('$\Rightarrow$') means local uniform convergence in $\mathbb{C}$ with respect to the spherical metric (Euclidian metric) of a sequence of meromorphic (holomorphic) functions.

This lemma was generalized by X.C pang as follows.

Pang-Zalcman Lemma. *(*[8, Lemma 2],[9, Theorem 1]*)*

Given a family $\mathcal{F}$ of functions meromorphic on the unit disk Δ which is not normal, then for every $-1 < \alpha < 1$, there exist
(a) a number $0 < r < 1$;
(b) points z_n, $|z_n| < r$ for every n;
(c) functions $f_n \in \mathcal{F}$; and
(d) positive numbers $\rho_n \to 0^+$,
such that

$$\frac{f_n(z_n + \rho_n\zeta)}{\rho_n^{\alpha}} \overset{\chi}{\Rightarrow} g(\zeta) \ ,$$

where g is a non-constant function in $\mathbb{C}$. Morever, g can be taken to satisfy the normalization $g^{\#}(\zeta) \le g^{\#}(0) = 1$, $\zeta \in \mathbb{C}$.

The case $\alpha = 0$ gives Zalcman's Lemma. These two lemmas have a local version that can be written uniformaly as:

Local Pang-Zalcman Lemma. (LPZ Lemma) cf. [11, Lemma 1.5], [5, Lemma 4.1].

A family $\mathcal{F}$ of functions meromorphic in a domain $D \subset \mathbb{C}$ is not normal at $z_0 \in D$ if and only if for every $-1 < \alpha < 1$ there exist
a) points $\{z_n\}_{n=1}^{\infty}$, $z_n \to z_0$;
b) functoins $\{f_n\}_{n=1}^{\infty} \in \mathcal{F}$;
c) positive numbers $\rho_n \to 0^+$;
such that

$$\rho_n^{-\alpha} f_n(z_n + \rho_n \zeta) \overset{\chi}{\Rightarrow} g(\zeta) \ , \tag{1.1}$$

where g is a nonconstant meromorphic function on $\mathbb{C}$, such that for every $\zeta \in \mathbb{C}$,

$$g^{\#}(\zeta) \le g^{\#}(0) = 1 \ . \tag{1.2}$$

The Pang-Zalcman Lemma and the LPZ Lemma also have extensions in case where we know that the multiplicities of the zeros (or of the poles) of members of the family of functions $\mathcal{F}$ are large enough (see [10, Lemma 2],[3, Lemma 3.2]). In this paper we shall not deal with these extensions, although our particular results are valid also for these extensions.

For a nonconstant function $\mathcal{F}$ meromorphic on $\mathbb{C}$, let $\mathcal{F}(f)$ be the non-normal family in $\mathbb{C}$

$$\mathcal{F}(f) = \{f(nz) : n \in \mathbb{N}\} \, .$$

Normality properties of the family $\mathcal{F}(f)$ has already been studied from various directions. Montel [4, PP. 158-176] was probably the first to deal with this topic. This subject was also studied in [6], [7] and [2].

The family $\mathcal{F}(f)$ is not normal in $\mathbb{C}$, and specifically is never normal at $z = 0$. Given a point z_0 where $\mathcal{F}(f)$ is not normal and $-1 < \alpha < 1$, then LPZ Lemma guarantees the existence of at least one function $g(\zeta)$, not constant and meromorphic on $\mathbb{C}$ that is obtained by the convergence process (1.1) described in this lemma. For a certain $-1 < \alpha < 1$, let $\Pi_\alpha(f)$ denote the collection of **all** the non-constant limit meromorphic functions $g(\zeta)$ (on $\mathbb{C}$) that are created in the convergence process (1.1) (but not necessarily satisfies the normalization (1.2)), considering all the points $z_0 \in \mathbb{C}$ of non-normality of $\mathcal{F}(f)$. For such a function g, we have by the definition of $\mathcal{F}(f)$ and by the LPZ Lemma a sequence $\{k_n\}_{n=1}^{\infty}$, $k_n \in \mathbb{N}$, $k_n \to \infty$, points $z_n \to z_0$ and positive numbers $\rho_n \to 0^+$ such that

$$f_{n,\alpha}(\zeta) := \frac{f(k_n z_n + k_n \rho_n \zeta)}{\rho_n^\alpha} \overset{\chi}{\Rightarrow} g(\zeta)\ . \tag{1.3}$$

Our main goal in this paper is to calculate, for every $-1 < \alpha < 1$, the collection $\Pi_\alpha(f)$ for the function

$$f(z) = R(z)e^{P(z)}, \tag{1.4}$$

where $R(z) \not\equiv 0$ is a general rational function and $P(z)$ is a nonconstant polynomial.

Before we state our result we establish some notation: If z_0 is a zero (pole) of order k of a nonconstant meromorphic function $f(z)$, then $\tilde{f}_{z_0}(z) := \frac{f(z)}{(z-z_0)^k}$ ($\hat{f}_{z_0}(z) := f(z)(z-z_0)^k$). Also for $z_0 \in \mathbb{C}$ and $r > 0$, $\Delta(z_0, r) := \{|z - z_0| < r\}$, $\overline{\Delta}(z_0, r) := \{|z - z_0| \le r\}$, and for $\theta \in \mathbb{R}$, R_θ denotes the ray from the origin with argument θ.

Now we state our main theorem. (The formulation is not short, as the proof is fairly involved.)

Theorem 1. *Let $f(z) = R(z)e^{P(z)}$ be as in (1.4), where $P(z) = a_k(z-\alpha_1)...(z-\alpha_k)$ (the α_i's may occur with repititions), $a_k \neq 0$; $R(z) = \frac{P_1(z)}{P_2(z)}$ where $P_1(z) = (z-\gamma_1)^{l_1}...(z-\gamma_m)^{l_m}$, $P_2(z) = (z-\beta_1)^{j_1}...(z-\beta_l)^{j_l}$. We assume that $\gamma_1, \cdots, \gamma_m$; $\beta_1, \cdots, \beta_l$ are all distinct. Let $L_1 := |P_1| = l_1 + ... + l_m$, $L_2 := |P_2| = j_1 + ... + j_l$. Then for the various values of $-1 < \alpha < 1$, $\Pi_\alpha(f)$ is given as follows:*

I. $\boldsymbol{k = |P| = 1}$

If $\boldsymbol{\alpha = 0}$*, then*

$$\Pi_0(f) = \left\{k_0 e^{A_1\zeta} : k_0 \neq 0, \arg A_1 = \arg a_1\right\} \bigcup \{f(C_1 + C_2\zeta) : C_1 \in \mathbb{C}, C_2 > 0\}.$$

If $\boldsymbol{0 < \alpha < 1}$*, then*

$$\Pi_\alpha(f) = \left\{k_0 e^{A_1\zeta} : k_0 \neq 0, \arg A_1 = \arg a_1\right\} \bigcup \left\{e^{P(\gamma_i)}\tilde{R}_{\gamma_i}(\gamma_i)(A_1\zeta + A_0)^{l_i} : 1 \leq i \leq m, A_0 \in \mathbb{C}, A_1 > 0\right\}.$$

If $\boldsymbol{-1 < \alpha < 0}$*, then*

$$\Pi_\alpha(f) = \left\{k_0 e^{A_1\zeta} : k_0 \neq 0, \arg A_1 = \arg a_1\right\} \bigcup \left\{e^{P(\beta_i)}\hat{R}_{\beta_i}(\beta_i)(A_1\zeta + A_0)^{-j_i} : 1 \leq i \leq l, A_0 \in \mathbb{C},\ A_1 > 0\right\}.$$

II. $\boldsymbol{k \geq 2}$ *If* $\boldsymbol{\alpha = 0}$*, then*

$$\Pi_0(f) = \{f(C_1 + C_2\zeta) : C_1 \in \mathbb{C}, C_2 > 0\} \bigcup \left[\bigcup_{l=0}^{k-1}\left\{e^{A_1\zeta + A_0} : A_0 \in \mathbb{C}, \arg A_1 = \left(\pm\frac{\pi}{2}(k-1) + \arg a_k + (k-1)2\pi l\right)/k\right\}\right].$$

If $\boldsymbol{0 < \alpha < 1}$, *then*

for $k = 2$

$$\Pi_\alpha(f) = \Big[\bigcup_{i=1}^{m} \Big\{ e^{P(\gamma_i)} A(\zeta + C)^{l_i} : \arg A = \arg \tilde{R}_{\gamma_i}(\gamma_i), C \in \mathbb{C} \Big\} \Big] \bigcup$$
$$\Big\{ e^{A_0 + A_1\zeta} : A_0 \in \mathbb{C}, \frac{\pi}{4} + \frac{\arg a_2}{2} \leq \arg A_1 \leq \frac{3\pi}{4} + \frac{\arg a_2}{2} \quad or$$
$$\frac{5\pi}{4} + \frac{a_2}{2} \leq \arg A_1 \leq \frac{7\pi}{4} + \frac{\arg a_2}{2} \Big\} \ .$$

For $k \geq 3$

$$\Pi_\alpha(f) = \Big[\bigcup_{i=1}^{m} \Big\{ e^{P(\gamma_i)} A(\zeta + C)^{l_i} : \arg A = \arg \tilde{R}_{\gamma_i}(\gamma_i), C \in \mathbb{C} \Big\} \Big] \bigcup$$
$$\Big\{ e^{A_1\zeta + A_0} : A_0 \in \mathbb{C}, A_1 \neq 0 \Big\} \ .$$

If $\boldsymbol{-1 < \alpha < 0}$, *then*

for $k = 2$

$$\Pi_\alpha(f) = \Big[\bigcup_{i=1}^{l} \Big\{ e^{P(\beta_i)} A(\zeta + C)^{-j_i} : \arg A = \arg \hat{R}_{\beta_i}(\beta_i), C \in \mathbb{C} \Big\} \Big] \bigcup$$
$$\Big\{ e^{A_0 + A_1\zeta} : A_0 \in \mathbb{C}, -\frac{\pi}{4} + \frac{\arg a_2}{2} \leq \arg A_1 \leq \frac{\pi}{4} + \frac{\arg a_2}{2}$$
$$or \quad \frac{3\pi}{4} + \frac{\arg a_2}{2} \leq \arg A_1 \leq \frac{5\pi}{4} + \frac{\arg a_2}{2} \Big\} \ .$$

For $k \geq 3$

$$\Pi_\alpha(f) = \Big[\bigcup_{i=1}^{l} \Big\{ e^{P(\beta_i)} A(\zeta + C)^{-j_i} : \arg A = \arg \hat{R}_i(\beta_i), C \in \mathbb{C} \Big\} \Big] \bigcup$$
$$\Big\{ e^{A_0 + A_1\zeta} : A_0 \in \mathbb{C}, A_1 \neq 0 \Big\} \ .$$

Observe that in each of the three intervals $\alpha = 0$, $0 < \alpha < 1$ and $-1 < \alpha < 0$, $\Pi_\alpha(f)$ is independent of α.

The proof of Theorem 1 is similar to climbing a ladder with four steps where each step is more complicated then the former step. In the first

step we calculate $\Pi_\alpha(M)$ for a general monome, $M(z) = (z-\alpha)^k$. In the second step we find $\Pi_\alpha(P)$ where P is a general nonconstant polynomial. In step 3 we calculate $\Pi_\alpha(R)$, where R is a general nonconstant rational function, and finally in the fourth step we find $\Pi_\alpha(Re^P)$. In each step we rely on the results of the previous steps. The first three steps is the contents of section 2, the proof of Theorem 1 is actually the fourth step which we prove in section 3. We note that for a nonconstant rational function, $z_0 = 0$ is the only point of non-normality in $\mathbb{C}$, and this is the situation in the first three steps. For $f = Re^P$, the points of non-normality lies on few rays through the origin, as we will see in the sequel. Throughout the proof we often deal with the connections between $\{z_n\}$ and $\{\rho_n\}$ in the LPZ Lemma. We hope this will contribute to the better understanding of the potential of this somewhat obscure lemma. As it is always possible to move to convergent subsequences (in the extended sense), we shall always assume without loss of generality that the sequences $\{k_n z_n\}$, $\{k_n \rho_n\}$ from (1.3) converge (in the extended sense). This assumption also applies to other sequences of complex numbers involved in our calculations.

The importance of this paper, beyond the result obtained in Theorem 1, lies in the technique that we used. The possible connections between z_n and ρ_n in (1.1) were used to deduce the limit function g. We note that the Pang-Zalcman Lemma is a common tool to establish normality of families of meromorphic functions. However, the proof of this lemma does not give an explicit relation between z_n to ρ_n, because some unknown parameter is involved in this relation (see [8, Lemma 2], [9, Theorem 1]). Hence, in general there is some difficulty in determing the limit function g. We expect that the detailed calculation that given here will contribute and promote the study of this subject.

2. Calculating $\Pi_\alpha(M)$, $\Pi_\alpha(P)$ and $\Pi_\alpha(R)$

2.1. **First step: Calculating $\Pi_\alpha(M)$ where $M(z) = (z-\beta)^k$.** Let $-1 < \alpha < 1$ and assume that $M_{n,\alpha}(\zeta) \Rightarrow g(\zeta)$, (where g is a non-constant entire function). This means that

$$(k_n \rho_n^{1-\frac{\alpha}{k}} \zeta + \frac{k_n z_n - \beta}{\rho_n^{\frac{\alpha}{k}}})^k \Rightarrow g(\zeta)\ . \tag{2.1}$$

The left hand side of (2.1) has a single zero of multiplicity k in $\mathbb{C}$, and thus, it follows by Rouché's Theorem that $g(\zeta)$ is also a monome of degree k. There must be $0 < A < \infty$ and $C \in \mathbb{C}$, such that $k_n \rho_n^{1-\frac{\alpha}{k}} \to A$ and $\frac{k_n z_n - \beta}{\rho_n^{\frac{\alpha}{k}}} \to C$ and so $g(\zeta) = (A\zeta + C)^k$. Conversely, given $A > 0$ and $C \in \mathbb{C}$, we set

$$k_n = n, \quad \rho_n = (\frac{A}{n})^{\frac{k}{k-\alpha}}, \quad z_n = \frac{A^{\frac{\alpha}{k-\alpha}} C + \beta n^{\frac{\alpha}{k-\alpha}}}{n^{1+\frac{\alpha}{k-\alpha}}} \tag{2.2}$$

to get (for every n) $M_n(\zeta) = (A\zeta + C)^k$. Thus, for every $-1 < \alpha < 1$

$$\Pi_\alpha(M) = \left\{ (A\zeta + C)^k : A > 0, C \in \mathbb{C} \right\}\ . \tag{2.3}$$

2.2. **Second step: Calculating $\Pi_\alpha(P)$ for a nonconstant polynomial $P(z)$.** Let $P(z) = L(z-\gamma_1)^{l_1}...(z-\gamma_m)^{l_m}$, $\gamma_i \neq \gamma_j$, $i \neq j$, $k := l_1 + l_2 ... + l_m$. Assume first that $\alpha = 0$ and that

$$P_{n,0}(\zeta) = P(k_n\rho_n\zeta + k_n z_n) \Rightarrow g(\zeta) \ . \tag{2.4}$$

By substituting $\zeta = 0$ in (2.4), we get that $\{k_n z_n\}$ is bounded and thus $k_n z_n \to C \in \mathbb{C}$ (recall that we always assume without loss of generality that $\{k_n z_n\}$, $\{k_n\rho_n\}$, etc. converge). Now, if $k_n\rho_n \to 0$ then g is constant and in case that $k_n\rho_n \to \infty$ then $g(\zeta) = \infty$ for every $\zeta \neq 0$. Hence $k_n\rho_n \to A$, $0 < A < \infty$ and we have $g(\zeta) = P(A\zeta + C)$.

On the other hand, given $0 < A < \infty$ and $C \in \mathbb{C}$, the trivial setting $k_n = n$, $\rho_n = \frac{A}{n}$, $z_n = \frac{C}{n}$ gives $P_{n,0}(\zeta) = P(A\zeta + C)$ and we get

$$\Pi_0(P) = \{P(A\zeta + C) : A > 0, C \in \mathbb{C}\} \, . \tag{2.5}$$

Consider now the case where $0 < \alpha < 1$. Here $P_{n,\alpha}(\zeta) \Rightarrow g(\zeta)$ means

$$\frac{L(k_n\rho_n\zeta + k_n z_n - \gamma_1)^{l_1}...(k_n\rho_n\zeta + k_n z_n - \gamma_m)^{l_m}}{\rho_n^\alpha} \Rightarrow g(\zeta) \ . \tag{2.6}$$

Because of $\rho_n^\alpha \to 0$, then by substituting $\zeta = 0$ in (2.6), we get that there exists $1 \leq i \leq m$ such that $k_n z_n \to \gamma_i$, since otherwise $P_{n,\alpha}(0) \to \infty$, and this would be a contradiction.

Without loss of generality, we assume that $i = 1$.

Claim 2.1. $k_n\rho_n \to 0$.

Proof. Indeed, if $k_n\rho_n \to \infty$, then for every $\zeta \neq 0$, $P_{n,\alpha}(\zeta) \to \infty$, a contradiction.

If $k_n\rho_n \to A$, $0 < A < \infty$, then there are some $R > 0$ and $N_0 \in \mathbb{N}$ such that for every ζ, $|\zeta| > R$, $n > N_0$ and $1 \leq i \leq m$, $|k_n\rho_n\zeta + k_n z_n - \gamma_i| \geq 1$ and thus $P_{n,\alpha}(\zeta) \to \infty$, a contradiction and the claim is proved. $\square$

We then get from (2.6) that

$$L\frac{(k_nz_n-\gamma_1+k_n\rho_n\zeta)^{l_1}}{\rho_n^\alpha}(\gamma_1-\gamma_2)^{l_2}(\gamma_1-\gamma_3)^{l_3}...(\gamma_1-\gamma_m)^{l_m}\Rightarrow g(\zeta)\ .$$

From the result in section 2.1 we then get that $g(\zeta)=\tilde{P}_{\gamma_1}(\gamma_1)(A\zeta+C)^{l_1}$ where $A>0$ and $C\in\mathbb{C}$.

Conversely, given $A>0$ and $C\in\mathbb{C}$, an analogous setting to (2.2)

$$k_n=n,\ \rho_n=(\tfrac{A}{n})^{\frac{l_1}{l_1-\alpha}},\ z_n=\frac{A^{\frac{\alpha}{l_1-\alpha}}C+\gamma_1n^{\frac{\alpha}{l_1-\alpha}}}{n^{1+\frac{\alpha}{l_1-\alpha}}}$$

gives

$$P_{n,\alpha}(\zeta)\Rightarrow\tilde{P_{\gamma_1}}(\gamma_1)(A\zeta+C)^{l_1}\ .$$

Observe that since $0<\alpha<1$, indeed $n\rho_n\to0$. Running over all the roots γ_i, $1\le i\le m$, of $P(z)$ we get that

$$\Pi_\alpha(P)=\left\{\tilde{P}_{\gamma_i}(\gamma_i)(A\zeta+C)^{l_i}:A>0,\ C\in\mathbb{C},\ 1\le i\le m\right\}\ . \tag{2.7}$$

We turn now to the case $-1<\alpha<0$. Suppose that

$$P_{n,\alpha}(\zeta)\Rightarrow g(\zeta)\ . \tag{2.8}$$

Claim 2.2. $k_n\rho_n\to\infty$.

Proof. If to the contrary, $k_n\rho_n\to A$, $A<\infty$ and $k_nz_n\to C\in\mathbb{C}$, then $P_{n,\alpha}(\zeta)\to0$ for every $\zeta\in\mathbb{C}$ and this is of course a contradiction. If $k_n\rho_n\to A<\infty$ and $k_nz_n\to\infty$ then (2.8) gives

$$L\frac{(k_nz_n)^k}{\rho_n^\alpha}\underbrace{\left[1+\frac{k_n\rho_n\zeta-\gamma_1}{k_nz_n}\right]^{l_1}...\left[1+\frac{k_n\rho_n\zeta-\gamma_m}{k_nz_n}\right]^{l_m}}_{T_n(\zeta)}\Rightarrow g(\zeta)\ . \tag{2.9}$$

Since

$$T_n(\zeta)\Rightarrow1\ ,$$

we get that $L\frac{(k_nz_n)^k}{\rho_n^\alpha}\Rightarrow g(\zeta)$ and we get that g is a constant, a contradiction. ∎

Claim 2.3. $\frac{z_n}{\rho_n}\to B\in\mathbb{C}$ *(equivalenty, for every* $1\le i\le m$, $\frac{k_nz_n-\gamma_i}{k_n\rho_n}\to B$*).*

Proof. If this were not the case, then for every $1 \le i \le m$, $\frac{k_n\rho_n}{k_n z_n - \gamma_i} \to 0$, and then

$$P_{n,\alpha}(\zeta) = \frac{L}{\rho_n^\alpha}\left[\prod_{i=1}^m (k_n z_n - \gamma_i)^{l_i}\right] \cdot \underbrace{\left[1 + \frac{k_n\rho_n}{k_n z_n - \gamma_1}\zeta\right]^{l_1} \cdots \left[1 + \frac{k_n\rho_n}{k_n z_n - \gamma_m}\zeta\right]^{l_m}}_{S_n(\zeta)}$$

$$\Rightarrow g(\zeta) \ .$$

Here also $S_n(\zeta) \Rightarrow 1$ and as in Claim 2.2, we get a contradiction and Claim 2.3 is proven. $\square$

We can write (2.8) as

$$L\frac{(k_n\rho_n)^k}{(\rho_n^\alpha)} \underbrace{\left[\zeta + \frac{k_n z_n - \gamma_1}{k_n\rho_n}\right]^{l_1} \cdots \left[\zeta + \frac{k_n z_n - \gamma_m}{k_n\rho_n}\right]^{l_m}}_{R_n(\zeta)} \Rightarrow g(\zeta) \ , \tag{2.10}$$

and since $R_n(\zeta) \Rightarrow (\zeta + B)^k$, we have $\frac{(k_n\rho_n)}{\rho_n{}^{\alpha/k}} \to A$, $0 < A < \infty$. Thus $g(\zeta) = L(A\zeta + C)^k$, where $C = AB$.

Conversely, let $g(\zeta) = L(A\zeta + C)^k$ where $A > 0$, $C \in \mathbb{C}$. We set $k_n = n$ and consider (2.10), we wish that $A = \frac{n\rho_n}{\rho_n^{\frac{\alpha}{k}}}$ and $\frac{z_n}{\rho_n} = \frac{C}{A}$. These requirements are fulfiled by the setting $\rho_n := (\frac{A}{n})^{\frac{k}{k-\alpha}}$, $z_n := \frac{C}{A}(\frac{A}{n})^{\frac{k}{k-\alpha}}$. Hence we get that for $-1 < \alpha < 0$

$$\Pi_\alpha(P) = \left\{L(A\zeta + C)^k : A > 0,\ C \in \mathbb{C}\right\} \ . \tag{2.11}$$

2.3. **Third step: Calculating** $\Pi_\alpha(R)$ **for a rational function** $R(z)$**.**

I. We assume first that R has at least one zero and one pole in $\mathbb{C}$. Denote

(2.12)

$$R(z) = L\frac{(z-\gamma_1)^{l_1}...(z-\gamma_m)^{l_m}}{(z-\beta_1)^{j_1}...(z-\beta_l)^{j_l}},\ k = l_1+...+l_m > 0,\ j = j_1+...+j_l > 0.$$

We assume that for some $-1 < \alpha < 1$

$$R_{n,\alpha}(\zeta) \overset{\chi}{\Rightarrow} g(\zeta) \ . \tag{2.13}$$

Observe first that Picard's great theorem and Rouché's Theorem imply that $\Pi_\alpha(R)$ contains only rational functions. We separate into subcases according to the value of α.

Case (A): $0 < \alpha < 1$.

Let us assume first that $k_n\rho_n \to C$, $0 < C < \infty$. In such case, if $k_n z_n \to \infty$, then as in (2.9) we deduce that g is a constant, a contradiction. If there exists some $b \in \mathbb{C}$ such that $k_n z_n \to b$, then by (2.13) (observe that $\rho_n^\alpha \to 0$) we get for every $0 \le \theta < 2\pi$, except finitely many θ's, that $R_{n,\alpha}(\zeta) \to \infty$ for every $\zeta = re^{i\theta}$, $r > 0$. This is a contradiction.

Secondly, we assume that $k_n\rho_n \to 0$. In such a situation if $k_n z_n \to \infty$ then $g(\zeta) \equiv d$ where d is some finite constant or $d \equiv \infty$, a contradiction. If $k_n z_n \to \eta$, $\eta \in \mathbb{C}$, then if for every i, j $\eta \neq \gamma_i, \beta_j$ then $g \equiv \infty$, a contradiction.

If $\eta = \beta_{i_0}$ for some j_0, $1 \le j_0 \le l$, then also by (2.13) $g \equiv \infty$, a contradiction.

If $\eta = \gamma_{i_0}$ for some $1 \le i_0 \le m$, then assume without loss of generality that $\eta = \gamma_1$. Then (2.13) can be written as

$$\frac{1}{\rho_n^\alpha}(k_n z_n - \gamma_1 + k_n\rho_n\zeta)^{l_1}\tilde{R}_{\gamma_1}(k_n z_n + k_n\rho_n\zeta) \overset{\chi}{\Rightarrow} g(\zeta) \ ,$$

and since

$$\tilde{R}_{\gamma_1}(k_n z_n + k_n\rho_n\zeta) \overset{\chi}{\Rightarrow} \tilde{R}_{\gamma_1}(\gamma_1) \ ,$$

we get by the case of a monome that

$$g(\zeta) = \tilde{R}_{\gamma_1}(\gamma_1)(A\zeta + C)^{l_1} \quad A > 0, \ C \in \mathbb{C} \ . \tag{2.14}$$

As in section 2.1, it can easily be shown that every function of the form (2.14) is in $\Pi_\alpha(R)$. Recall now that C_0 can be any value $1 \le i_0 \le m$,

and we get that the contribution to $\Pi_\alpha(R)$ from this possibility is

$$\left\{g(\zeta)=\tilde{R}_{\gamma_i}(\gamma_i)(A\zeta+C)^{l_i}:\quad C\in\mathbb{C},\ A>0,\ 1\le i\le m\right\}. \tag{2.15}$$

The last option in case (A) is that $k_n\rho_n\to\infty$. Similarly to the case $k_n\rho_n\to 0$, we deduce that $\frac{z_n}{\rho_n}\to C$, $C\in\mathbb{C}$. (Recall that we can assume with no loss of generality that sequences as $\left\{\frac{z_n}{\rho_n}\right\}$ converges in the extended sense.) We can write

$$R_{n,\alpha}(\zeta)=L\frac{(k_n\rho_n)^{l_1+\cdots+l_m}(\zeta+\frac{k_nz_n-\gamma_1}{k_n\rho_n})^{l_1}\dots(\zeta+\frac{k_nz_n-\gamma_m}{k_n\rho_n})^{l_m}}{\rho_n^\alpha(k_n\rho_n)^{j_1+\cdots+j_l}(\zeta+\frac{k_nz_n-\beta_1}{k_n\rho_n})^{j_1}\dots(\zeta+\frac{k_nz_n-\beta_l}{k_n\rho_n})^{j_l}}$$

$$=L\frac{(k_n\rho_n)^{k}(\zeta+\frac{k_nz_n-\gamma_1}{k_n\rho_n})^{l_1}\dots(\zeta+\frac{k_nz_n-\gamma_m}{k_n\rho_n})^{l_m}}{\rho_n^\alpha(k_n\rho_n)^{j}(\zeta+\frac{k_nz_n-\beta_1}{k_n\rho_n})^{j_1}\dots(\zeta+\frac{k_nz_n-\beta_l}{k_n\rho_n})^{j_l}}$$

$$=L\frac{(k_n\rho_n)^{k-j}(\zeta+\frac{k_nz_n-\gamma_1}{k_n\rho_n})^{l_1}\dots(\zeta+\frac{k_nz_n-\gamma_m}{k_n\rho_n})^{l_m}}{\rho_n^\alpha(\zeta+\frac{k_nz_n-\beta_1}{k_n\rho_n})^{j_1}\dots(\zeta+\frac{k_nz_n-\beta_l}{k_n\rho_n})^{j_l}}\ .$$

Observe that for every i and j, $\frac{k_nz_n-\gamma_i}{k_n\rho_n},\frac{k_nz_n-\beta_j}{k_n\rho_n}\to C$. Thus, if $k\ge j$ this is a contradiction, since the only candidate to be a limit function is $g\equiv\infty$.

If $k<j$, then $L_0:=\lim\frac{(k_n\rho_n)^{k-j}}{\rho_n^\alpha}$ must satisfy $L_0\ne 0,\infty$, since otherwise $g\equiv 0$ or $g\equiv\infty$, as the value of L_0. We deduce that $g(\zeta)=L\cdot L_0(\zeta+C)^{k-j}$. But $R_{n,\alpha}(\zeta)$ vanishes at $\frac{k_nz_n-\gamma_1}{k_n\rho_n},\dots,\frac{k_nz_n-\gamma_m}{k_n\rho_n}$ and thus $g(-C)=0$, a contradiction. Hence the collection (2.15) is $\Pi_\alpha(R)$.

Case (B): $-1<\alpha<0$.

The calculation of $\Pi_\alpha(R)$ is immediate since $R_{n,\alpha}(\zeta)\overset{\chi}{\Rightarrow}g(\zeta)$ in $\mathbb{C}$ if and only if $(\frac{1}{R})_{n,-\alpha}(\zeta)\overset{\chi}{\Rightarrow}\frac{1}{g}(\zeta)$ in $\mathbb{C}$, and since $0<-\alpha<1$. Thus, by Case (A), $\Pi_\alpha(R)=\left\{\hat{R}_{\beta_n}(\beta_n)((A\zeta+C)^{j_n})^{-1}:A>0,\ C\in\mathbb{C},\ 1\le n\le l\right\}$.

Case (C): $\alpha=0$.

Assume first that $k_n\rho_n \to 0$. Then if $k_n z_n \to \infty$ we deduce that $g \equiv c$, $c \in \mathbb{C}$, a contradiction.

If $k_n z_n \to b$, $b \in \mathbb{C}$, then in case $b \neq \gamma_i, \beta_j$ for every i, j we get by (2.13) that g is constant, a contradiction.

If $b = \gamma_{i_0}$, $1 \leq i_0 \leq m$, then $g \equiv 0$, a contradiction. If $b = \beta_{j_0}$, $1 \leq j_0 \leq l$ then $g \equiv \infty$, a contradiction.

The next possibility we examine is $k_n\rho_n \to \infty$. As in Case (A) or Case (B) we must have $\frac{z_n}{\rho_n} \to c \in \mathbb{C}$. Then we can write

$$R_{n,0}(\zeta) = L(k_n\rho_n)^{k-j}\frac{(\zeta + \frac{k_n z_n - \gamma_1}{k_n\rho_n})^{l_1}\dots(\zeta + \frac{k_n z_n - \gamma_m}{k_n\rho_n})^{l_m}}{(\zeta + \frac{k_n z_n - \beta_1}{k_n\rho_n})^{j_1}\dots(\zeta + \frac{k_n z_n - \beta_l}{k_n\rho_n})^{j_l}}\,.$$

In any of the cases $k = j$, $k > j$ or $k < j$, we get a contradiction. So it must be the case $k_n\rho_n \to c$, $0 < c < \infty$. Then, if $k_n z_n \to \infty$ then similarly to the case $k_n\rho_n \to 0$, we get that g is constant so $k_n z_n \to b$, $b \in \mathbb{C}$ and $g(\zeta) = R(b + c\zeta)$.

Conversely, for every $b \in \mathbb{C}$, $c > 0$, we can take $k_n = n$, $\rho_n = \frac{c}{n}$, $z_n = \frac{b}{n}$ to get $R_{n,0}(\zeta) \overset{\chi}{\Rightarrow} R(b+c\zeta)$ in $\mathbb{C}$, so $\Pi_0(R) = \{R(b + c\zeta) : b \in \mathbb{C}, c > 0\}$.

II. Now we consider the case where $R(z)$ has only zeros or only poles. If $R(z)$ has only zeros, then R is a polynomial and this case was discussed in section 2.2. If $R(z)$ has only poles then $R = \frac{1}{P}$ where P is a polynomial, and we can use the same principle as in Case (B) of (I) of the present subsection, and then deduce by the results in section 2.2 (see (2.5), (2.7) and (2.11)) the following:

For $\alpha = 0$ we get by (2.5)

$$\Pi_0(R) = \{R(A\zeta + C) : A > 0, C \in \mathbb{C}\}\,.$$

For $0 < \alpha < 1$ we get by (2.11)

$$\Pi_\alpha(R) = \left\{\frac{L}{(A\zeta + C)^j} : A > 0, C \in \mathbb{C}\right\}\,.$$

And for $-1<\alpha<0$ we have by (2.7)

$$\Pi_\alpha(R)=\left\{\hat{R}_{\beta_n}(\beta_n)((A\zeta+C)^{j_n})^{-1} : A>0,\ C\in\mathbb{C},\ 1\le n\le l\right\} .$$

3. Finding $\Pi_\alpha(Re^P)$

Let $f(z)=R(z)e^{P(z)}$ where

$$R=\frac{P_1}{P_2},\ P_1(z):=(z-\gamma_1)^{l_1}\dots(z-\gamma_m)^{l_m}, \tag{3.1}$$

$$P_2(z):=(z-\beta_1)^{j_1}\cdots(z-\beta_l)^{j_l},\ L_1:=\big|P_1\big|=l_1+\cdots+l_m;$$

$$L_2:=\big|P_2\big|=j_1+\cdots+j_l,\ L_1,L_2\ge 0,\ L\neq 0\ .$$

The case $R=L_1\frac{P_1}{P_2}$, $L\neq 0,1$ is also included here, i.e., we can assume that $L=1$, since otherwise $L=e^{a_0'}$, $a_0'\neq 0$. We can write $\hat{a}_0=a_0+a_0'$ instead of a_0 as the constant coefficient of $P(z)$.

Also let us denote $P(z)=a_kz^k+a_{k-1}z^{k-1}+\cdots+a_0$, $a_k\neq 0$. We wish to find $\Pi_\alpha(f)$ for $-1<\alpha<1$, but first we need some preparation.

3.1. **Auxiliary lemmas and a remark.**

Lemma 3.1. *Let f be a nonconstant meromorphic function in $\mathbb{C}$ and $-1<\alpha<1$. Then*

(1) *If $g(\zeta)\in\Pi_\alpha(f)$ then for every $C\in\mathbb{C}$ $g(\zeta+C)\in\Pi_\alpha(f)$ and*

(2) *If $e^{a\zeta+b}\in\Pi_\alpha(f)$ then for every $a'\neq 0$ such that $\arg(a')=\arg(a)$ and for every $b'\in\mathbb{C}$, $e^{a'\zeta+b'}\in\Pi_\alpha(f)$.*

Proof. Suppose that $g\in\Pi_\alpha(f)$, then we have $\frac{f(k_nz_n+k_n\rho_n(\zeta+C))}{\rho_n^\alpha}\overset{\chi}{\Rightarrow} g(\zeta)$ in $\mathbb{C}$, with $\rho_n\to 0^+$, $z_n\to z_0$ and $k_n\in\mathbb{N}$. We set $\rho_n'=\rho_n$, $z_n'=z_n+\rho_nC\to z_0$ and get

$$\frac{f(k_nz_n'+k_n\rho_n'\zeta)}{{\rho'}_n^\alpha}=\frac{f(k_nz_n+k_n\rho_n(\zeta+C))}{\rho_n^\alpha}\overset{\chi}{\Rightarrow} g(\zeta+C)\ ,$$

and this proves (1). For the proof of (2) assume that $\frac{f(k_nz_n+k_n\rho_n\zeta)}{\rho_n^\alpha} \overset{\chi}{\Rightarrow} e^{a\zeta+b}$ in $\mathbb{C}$. Define for a' with $\arg a' = \arg a$, $\rho_n' = \frac{a'}{a}\rho_n \to 0^+$ and $(\frac{A'}{A})^{-\alpha} = e^{b_0}$, where $b_0 \in \mathbb{R}$. We have

$$\frac{f(k_nz_n + k_n\rho_n'\zeta)}{(\rho_n')^\alpha} = \frac{f(k_nz_n + k_n\rho_n(\frac{a'}{a}\zeta))}{\rho_n^\alpha(a'/a)^\alpha} \overset{\chi}{\Rightarrow} g(\frac{a'}{a}\zeta)e^{b_0} = e^{a'\zeta+b+b_0}\ .$$

By (1) we can replace $b + b_0$ with every $b' \in \mathbb{C}$. This completes the proof of the lemma. $\square$

Remark. *Let F be a family of non-vanishing holomorphic functions which is not normal at z_0 and let $-1 < \alpha < 1$. Then the convergence process (1.1) in the LPZ Lemma guarantees a limit function $g(\zeta)$ with $g^\#(\zeta) \le 1$ for every $\zeta \in \mathbb{C}$. By a theorem of Clunie and Hayman* [1, Theorem 3]*, the order of g is at most 1 and since $g(\zeta) \ne 0$, $\zeta \in \mathbb{C}$, by Hurwitz's Theorem we deduce that $g(\zeta) = e^{a\zeta+b}$. The results which we will prove in the detailed process of calculating $\Pi_\alpha(Re^P)$ are indeed consistent with this theorem of Clunie and Hayman.*

Lemma 3.2. *Let $f = Re^P$ be given by (3.1). Then the points where $F(f)$ is not normal in $\mathbb{C}$ are exactly*

$$\left\{\bigcup_{l=0}^{k-1} R_{\theta_k^+(l)}\right\} \bigcup \left\{\bigcup_{l=0}^{k-1} R_{\theta_k^-(l)}\right\} \tag{3.2}$$

where for every $0 \le l \le k-1$, $\theta_k^+(l)$ and $\theta_k^-(l)$ are defined by $\theta_k^\pm(l) = \frac{\pm\frac{\pi}{2} - \arg a_k}{k} + \frac{2\pi l}{k}$ and $\arg a_k$ is taken to be in $[0, 2\pi)$.

Observe that for every $0 \le l \ne j \le k-1$, $\theta_k^\pm(l) \ne \theta_j^\pm(l)$.

Proof. For every $z_0 \ne 0$ that is not in the union (3.2) there exist $r > 0$ and $0 \le l \le k-1$ such that

$$\overline{\Delta}(z_0, r) \subset S\left(\frac{\theta_k^+(l) + \theta_k^-(l+1)}{2}, \frac{\theta_k^+(l) - \theta_k^-(l+1)}{2}\right) \tag{3.3}$$

or that

$$\overline{\Delta}(z_0, r) \subset S\left(\frac{\theta_k^-(l) + \theta_k^+(l)}{2}, \frac{\theta_k^+(l) - \theta_k^-(l)}{2}\right). \tag{3.4}$$

There is some small $\varepsilon_0 > 0$ such that in the case that (3.3) holds, then for every $z \in \Delta(z_0, r)$ and for every $n \in \mathbb{N}$

$$\pi/2 + 2\pi l + \varepsilon_0 < \arg a_k(nz)^k < 3\pi/2 + 2\pi l - \varepsilon_0.$$

In the case (3.4), then for every $z \in \Delta(z_0, r)$

$$-\pi/2 + 2\pi l + \varepsilon_0 < \arg\left(a_k(nz)^k\right) < \pi/2 + 2\pi l - \varepsilon_0.$$

Hence there exists N_0, such that if $n > N_0$ and $z \in \Delta(z_0, r)$, then

$$\pi/2 + 2\pi l + \varepsilon_0/2 < \arg P(nz) < 3\pi/2 + 2\pi l - \varepsilon_0/2$$

in the case of (3.4).

Hence in the case of (3.3) $f(nz) \to 0$ uniformly in $\Delta(z_0, r)$ and in case of (3.4) $f(nz) \to \infty$ uniformly in $\Delta(z_0, r)$, that is, in any case $F(f)$ is normal at z_0.

If z_0 belongs to one of the $2k$ rays from the union (3.2), then any neighbourhood of z_0 contains points z where $f(nz) \to 0$ and points z where $f(nz) \to \infty$. So $F(f)$ is not normal at z_0. $\square$

We are now ready to calculate $\Pi_\alpha(f)$. We shall do this by separating into 2 cases according to the value of $k = |P|$.

3.2. **Calculating $\Pi_\alpha(Re^P)$ for linear polynomial $P(z)$.** We have $P(z) = a_1 z + a_0$, $a_1 \neq 0$. Let z_0 be a point where $F(f)$ is not normal. We assume thar for some $-1 < \alpha < 1$

$$f_{n,\alpha}(\zeta) = \frac{f(k_n z_n + k_n \rho_n \zeta)}{\rho_n^\alpha} \overset{\chi}{\Rightarrow} g(\zeta) \ , \tag{3.5}$$

where $z_n \to z_0$, $\rho_n \to 0^+$ and $k_n \to \infty$.

Case (A) $z_0 \neq 0$.

In this case

$$k_n z_n \to \infty \quad \text{and} \quad \frac{z_n}{\rho_n} \to \infty \ , \tag{3.6}$$

and thus

$$\frac{R(k_nz_n + k_n\rho_n\zeta)}{(k_nz_n)^{L_1-L_2}} \Rightarrow 1 \ .$$

We deduce that

$$\tilde{g}_n(\zeta) := (k_nz_n)^{L_1-l_2}\frac{e^{a_1k_nz_n+a_0}e^{a_1k_n\rho_n\zeta}}{\rho_n^{\alpha}} \Rightarrow g(\zeta) \ . \tag{3.7}$$

Since $\tilde{g}_n(\zeta) \neq 0$ for $\zeta \in \mathbb{C}$, we deduce that $g \neq 0$ in $\mathbb{C}$, i.e., $g = e^Q$ where Q is an entire function. With a suitable branch of the logarithm, we have

$$e^{a_1k_nz_n+a_0-\alpha\ln\rho_n+(L_1-L_2)\log k_nz_n+a_1k_n\rho_n\zeta} \Rightarrow e^{Q(\zeta)} \ .$$

Thus, there are integers m_n such that

$$a_1k_nz_n + a_0 - \alpha\ln\rho_n + (L_1-L_2)\log k_nz_n + a_1k_n\rho_n\zeta + 2\pi i m_n \Rightarrow Q(\zeta) \ .$$

Hence Q is a linear function, $Q(\zeta) = A_1\zeta + A_0$ and $g(\zeta) = e^{A_0}\cdot e^{A_1\zeta}$. Substituting $\zeta = 0$ in (3.7) gives that

$$\frac{(k_nz_n)^{L_1-L_2}e^{a_1k_nz_n+a_0}}{\rho_n^{\alpha}} \underset{n\to\infty}{\to} e^{A_0} \ ,$$

and thus

$$a_1k_n\rho_n \to A_1 \tag{3.8}$$

and $\arg A_1 = \arg a_1$. By (1) and (2) of Lemma 3.1 we deduce that the contribution of $z_0 \neq 0$, point of non-normality of $F(f)$ to $\Pi_\alpha(f)$, is

$$\left\{k_0e^{A_1\zeta} : k_0 \neq 0, \ \arg A_1 = \arg a_1\right\} \ . \tag{3.9}$$

Observe that this collection is independent of α.

Case (B) $z_0 = 0$.

We separate into subcases according to the behaviour of $\{k_nz_n\}$.

$\boldsymbol{k_nz_n \to b}$, $\boldsymbol{b} \in \mathbb{C}$.

In this case, if $k_n\rho_n \to \infty$, then when $\alpha \leq 0$ it holds for every $\zeta \neq 0$, $\zeta \in R_\theta$ and $\theta_1^-(0) < \theta < \theta_1^+(0)$, that $f_{n,\alpha}(\zeta) \underset{n\to\infty}{\to} \infty$, and this implies

that $g \equiv \infty$, a contradiction. If $\alpha \geq 0$, then for every $\zeta \neq 0$, $\zeta \in R_\theta$,

$$\theta_1^+(0) < \theta < \theta_1^-(1) \ , \tag{3.10}$$

we have $f_{n,\alpha}(\zeta) \to 0$ and this also leads to a contradiction.

If $k_n\rho_n \to a$, $a > 0$, then in case that $\alpha > 0$, it holds for every ζ such that $R(a\zeta + b) \neq 0$ that $g(\zeta) = \infty$, and this is impossible.

If $\alpha < 0$, then for every ζ such that $R(a\zeta + b) \neq \infty$, $g(\zeta) = 0$, again a contradiction.

So the case $k_n z_n \to b$, $k_n z_n \to a > 0$ can happen only with $\alpha = 0$, and indeed in this case the limit function is $g(\zeta) = f(a\zeta + b)$ and every such function is attained with $k_n = n$, $\rho_n = \frac{a}{n}$, $z_n = \frac{b}{n}$.

So this possibility gives the collection

$$\{f(a\zeta + b) : a > 0, b \in \mathbb{C}\} \tag{3.11}$$

to $\Pi_0(f)$.

We are left with the option $k_n\rho_n \to 0$. We then have that

$$R_{n,\alpha}(\zeta) = \frac{R(k_n z_n + k_n\rho_n\zeta)}{\rho_n^\alpha} \overset{\chi}{\Rightarrow} g(\zeta)e^{-P(b_1)}. \tag{3.12}$$

If $\alpha = 0$ then g is a constant, a contradiction. If $0 < \alpha < 1$, then in the case that $P_1(z)$ is a constant, $R_{n,\alpha}(\zeta) \Rightarrow \infty$ and $g \equiv \infty$, a contradiction. If P_1 is not a constant then necessarily there exists some $1 \leq i \leq m$ such that $k_n z_n \underset{n\to\infty}{\to} \gamma_i$. We then have

$$\frac{(k_n z_n - \gamma_i + k_n\rho_n\zeta)^{l_i}}{\rho_n^\alpha} \Rightarrow \frac{g(\zeta)e^{-P(\gamma_i)}}{\tilde{R}_{\gamma_i}(\gamma_i)} \ .$$

By the case of monome (see (2.3)), we get that

$$g(\zeta) = e^{P(\gamma_i)}\tilde{R}_{\gamma_i}(\gamma_i)(a\zeta + b)^{l_i}, \ b \in \mathbb{C}, \ a > 0 \tag{3.13}$$

and by the setting of (2.2), every $g(\zeta)$ of the form (3.13) belongs to $\Pi_\alpha(f)$ (corresponding to all the roots γ_i of $P_1(z)$, $1 \leq i \leq m$).

Now, if $-1 < \alpha < 0$ then $0 < -\alpha < 1$ and as in Case (B) of (I) in section 2.3, or in (II) in section 2.3, we get that if $P_2(z)$ is a constant

and then $g \equiv \infty$, a contradiction. If $P_2(z)$ is not a constant then $k_n z_n \underset{n\to\infty}{\to} \beta_i$ for some $1 \le i \le l$, and analogously to (3.13) we have

$$g(\zeta) = \frac{e^{P(\beta_i)}\hat{R}_{\beta_i}(\beta_i)}{(a\zeta+b)^{j_i}},\ a>0,\ b\in\mathbb{C}\ , \tag{3.14}$$

and conversely, every function $g(\zeta)$ as in (3.14), (corresponding to the various roots of $P_2(z)$, β_i, $1 \le i \le l$) belongs to $\Pi_\alpha(f)$.

We turn now to the second subcase of Case (B).

$\boldsymbol{k_n z_n \to \infty}$.

In this situation, if $k_n\rho_n \to \infty$ and $\frac{z_n}{\rho_n} \to \infty$ then (3.5) is equivalent to

$$\frac{(k_n z_n)^{l_1-l_2}}{(\rho_n^\alpha)} e^{a_1 k_n z_n + a_0} e^{a_1 k_n \rho_n \zeta} \Rightarrow g(\zeta)\ ,$$

and we deduce that we must have $g(\zeta) = k_0 e^{a\zeta}$.

On the other hand, for every ζ, $\zeta \notin R_{\theta_1^+(0)} \bigcup R_{\theta_1^-(0)}$, $g(\zeta) = 0$ or $g(\zeta) = \infty$, and this is a contradiction.

Suppose that $k_n\rho_n \to \infty$ and $\frac{z_n}{\rho_n} \to d$, $d \in \mathbb{C}$. Then (3.5) can be written as

$$f_{n,\alpha}(\zeta) = \frac{R[k_n\rho_n(\zeta + \frac{z_n}{\rho_n})]e^{a_0}e^{a_1 k_n \rho_n(\zeta+\frac{z_n}{\rho_n})}}{\rho_n^\alpha} \overset{\chi}{\Rightarrow} g(\zeta)\ . \tag{3.15}$$

When ζ belongs to the half plane $\{\zeta : -\pi/2 < \arg(a_1) + \arg(\zeta + C) < \pi/2\}$ we have $f_{n,\alpha}(\zeta) \to \infty$ if $\alpha \ge 0$, while if $\alpha \le 0$, then $f_{n,\alpha}(\zeta) \to 0$ for every ζ in the complementary half plane, $\{\zeta : \pi/2 < \arg(a_1) + \arg(\zeta + C) < 3\pi/2\}$, and we have got a contradiction.

To summarize, the possibility $k_n z_n \to \infty$ and $k_n\rho_n \to \infty$ does not occur.

Now if $k_n\rho_n \to a$, $a \in \mathbb{C}$ then (3.5) is equivalent to (3.7) and $g(\zeta) = e^{A+B\zeta}$ and it must be that $a > 0$ and $A = a \cdot a_1$.

In order to show that for each $B \in \mathbb{C}$ and for each A satisfying $\arg(A) = \arg(a_1)$, the function $g(\zeta) = e^{A\zeta+B}$ belongs to $\Pi_\alpha(f)$, it

is enough by Lemma 3.1 to show that one such function is attained (in fact, it is equally easy to show directly that each such function is attained).

Indeed, let us take a sequence of non-zero numbers, $z_0^{(l)} \underset{l\to\infty}{\to} 0$ such that for every $l \geq 1$, $\arg(z_0^{(l)}) = \pi/2 - \arg(a_1)$. By the results in Case (A) (see (3.9)), for every $l \geq 1$ there are sequences, $k_m^{(l)} \underset{m\to\infty}{\to} \infty$, $z_m^{(l)} \underset{m\to\infty}{\to} z_0^{(l)}$ and $\rho_m^{(l)} \underset{m\to\infty}{\to} 0^+$ such that

$$\frac{f(k_m^{(l)} z_m^{(l)} + k_m^{(l)} \rho_m^{(l)} \zeta)}{\rho_m^{(l)\alpha}} \overset{\chi}{\Rightarrow} e^{a_1 \zeta} .$$

Now for every $n \geq 1$, there is $m_n > n$ such that

$$\left|k_{m_n}^{(n)} \cdot z_{m_n}^{(n)}\right| > n, \quad \rho_{m_n}^{(n)} < \frac{1}{n} \quad \text{and} \quad \left|z_{m_n}^{(n)} - z_0^{(n)}\right| < \frac{1}{n} , \tag{3.16}$$

and such that

$$\max_{\zeta \leq n} \left| \frac{f(k_{m_n}^{(n)} z_{m_n}^{(n)} + k_m^{(n)} \rho_{m_n}^{(n)} \zeta)}{\rho_m^{(n)^\alpha}} - e^{a_1 \zeta} \right| \leq \frac{1}{n} .$$

We define now for every $n \geq 1$, $k_n := k_{m_n}^{(n)}$, $\rho_n := \rho_{m_n}^{(n)}$, $z_n := z_{m_n}^{(n)}$. By (3.16) we deduce that

$$\frac{f(k_n z_n + k_n \rho_n \zeta)}{\rho_n^\alpha} \overset{\chi}{\Rightarrow} e^{a_1 \zeta} ,$$

as required (with $k_n z_n \to \infty$ and $k_n \rho_n \to 1$) see (3.8).

Hence the collection of limit functions created by the possibility $k_n z_n \to \infty$ and $k_n z_n \to a$, $a \in \mathbb{C}$ is exactly

$$\left\{e^{A\zeta + B} : B \in \mathbb{C} \quad \text{and} \quad \arg A = \arg(A_1)\right\} . \tag{3.17}$$

We can now summarize the results and conclude the assertion of Theorem 1 for the case where P is linear.

For $\alpha = 0$, we get by (3.9), (3.11), and (3.17) (and the various contradictions along the way)

$$\Pi_0(f) = \left\{e^{a\zeta + b} : \arg a = \arg a_1, b \in \mathbb{C}\right\} \bigcup \left\{f(a\zeta + b) : a > 0,\ b \in \mathbb{C}\right\} .$$

For $0 < \alpha < 1$, (3.9), (3.13) and (3.17) give

$$\Pi_\alpha(f) = \left\{e^{a\zeta+b} : \arg a = \arg a_1,\ b \in \mathbb{C}\right\}$$

$$\bigcup \left\{e^{P(\gamma_i)}\tilde{R}_{\gamma_i}(\gamma_i)(a\zeta+b)^{l_i} : a > 0,\ b \in \mathbb{C},\ 1 \le i \le m\right\}.$$

For $-1 < \alpha < 0$ we have by (3.9), (3.14) and (3.17)

$$\Pi_\alpha(f) = \left\{e^{a\zeta+b} : \arg a = \arg a_1,\ b \in \mathbb{C}\right\}$$

$$\bigcup \left\{e^{P(\beta_i)}\hat{R}_{\beta_i}(\beta_i)/(a\zeta+b)^{j_i} : a > 0, b \in \mathbb{C}, 1 \le i \le l\right\}.$$

3.3. **Calculating $\Pi_\alpha(Re^P)$ when $k = |P| \ge 2$.** We consider (3.5) and separate into cases according the behaviour of $\{k_n z_n\}$.

Case (A) $\boldsymbol{k_n z_n \to b} \in \mathbb{C}$.

Of course in this case $z_n \to 0$.

If $k_n\rho_n \to \infty$, then if $\alpha \le 0$ it holds for every non-zero ζ, $\zeta \in R_\theta$, for $\theta_k^-(l) < \theta < \theta_k^+(l)$, $0 \le l \le k-1$, that $f_{n,\alpha}(\zeta) \underset{n\to\infty}{\to} \infty$ (compare (3.10)), and this is a contradiction. If $\alpha \ge 0$ then for every non-zero ζ, $\zeta \in R_\theta$, $\theta_k^+(l) < \theta < \theta_k^-(l+1)$, $f_{n,\alpha}(\zeta) \underset{n\to\infty}{\to} 0$, and this is a contradiction.

Hence we deduce that $k_n\rho_n \to a \in \mathbb{C}$.

If $a > 0$ and $\alpha \ne 0$, then similarly to the parallel case when $|P| = k = 1$ (Case (B) in section 3.2) we get a contradiction.

The possibility $a > 0$ and $\alpha = 0$, as in the case $|P| = 1$, gives the collection

$$\{f(a\zeta+b) : a > 0,\ b \in \mathbb{C}\} \tag{3.18}$$

to $\Pi_0(f)$.

We are left with the possibility $k_n\rho_n \to 0$. We then get that

$$\frac{R(k_n z_n + k_n\rho_n\zeta)}{\rho_n^\alpha} \overset{\chi}{\Rightarrow} g(\zeta)e^{-P(b)}\ ,$$

that is, $\tilde{g} := g \cdot e^{-P(b)}$ belongs to $\Pi_\alpha(R)$. Thus, in the case $0 < \alpha < 1$ we get by the discussion in section 2.3 that for some $1 \le i_0 \le m$, $b = \gamma_{i_0}$ (in case $\left|P_1\right| > 0$, otherwise we get a contradiction) and consider all γ_i,

$1 \le i \le m$, we get from (2.15) that the case $k_n\rho_n \to 0$, $k_n z_n \to b \in \mathbb{C}$ gives the collection

$$\bigcup_{i=1}^{m} \left\{ e^{P(\gamma_i)} \tilde{R}_{\gamma_i}(\gamma_i)(A_1\zeta + A_2)^{l_1},\ A_1 > 0,\ A_2 \in \mathbb{C} \right\} \tag{3.19}$$

to $\Pi_0(f)$.

In the case $-1 < \alpha < 0$ we get (similarly to the parallel subcase in Case (B) in Section 3.2) the collection

$$\bigcup_{i=1}^{l} \left\{ e^{P(\beta_i)} \hat{R}_{\beta_i}(\beta_i)(A_1\zeta + A_2)^{-j_1},\ A_1 > 0,\ A_2 \in \mathbb{C} \right\} . \tag{3.20}$$

The case $\alpha = 0$ leads to a contradiction, similarly to the parallel case in Case (B) in Section 3.2.

Case (B) $\boldsymbol{k_n z_n \to \infty}$

We have $z_n \to z_0$, and in this case both options $z_0 = 0$ or $z_0 \neq 0$ are possible. First we deal with the option $z_0 \neq 0$, i.e. $z_0 = re^{i\theta_0}$ where θ_0 is one of the arguments of the $2k$ rays from (3.2). Since $\frac{\rho_n}{z_n} \to 0$, then (3.5) is equivalent to

$$(k_n z_n)^{L_1 - L_2} \frac{e^{P(k_n z_n + k_n \rho_n \zeta)}}{\rho_n^{\alpha}} \overset{\chi}{\Rightarrow} g(\zeta) . \tag{3.21}$$

By Hurwitz's Theorem $g(\zeta) = e^{Q(\zeta)}$, where Q is an entire function. For a suitable branch of the logarithm, we have

$$e^{P(k_n z_n + k_n \rho_n \zeta) + (L_1 - L_2)\log k_n z_n - \alpha \ln \rho_n} \Rightarrow e^{Q(\zeta)} .$$

Thus, there exist integers $\{m_n\}$ such that

(3.22)

$$P(k_n z_n + k_n \rho_n \zeta) + (L_1 - L_2)\ln|k_n z_n| + i(L_1 - L_2)(\theta_0 + \varepsilon_n) - \alpha \ln \rho_n + 2\pi m_n$$

$$\Rightarrow Q(\zeta) ,$$

where $\varepsilon_n \in \mathbb{R}$, $\varepsilon_n \to 0$.

We conclude that Q is a polynomial of degree $|Q| \leq k$. Denote $Q(z) = A_0 + A_1\zeta + ... + A_k\zeta^k$. Comparing coefficients of the two sides of (3.22) gives the following relations

$$
\begin{aligned}
&a_k(k_n\rho_n)^k \underset{n\to\infty}{\to} A_k \\
&\quad\vdots \\
&a_k\left(k_n\rho_n\right)^{k-i}\begin{pmatrix} k \\ k-i \end{pmatrix}\left(k_n z_n\right)^i \to A_{k-i} \\
&\quad\vdots \\
&a_k\left(k_n\rho_n\right)\begin{pmatrix} k \\ 1 \end{pmatrix}\left(k_n z_n\right)^{k-1} \to A_1 \\
&a_k(k_n z_n-\alpha_1)...(k_n z_n-\alpha_k)+(L_1-L_2)\ln|k_n z_n| \\
&+i\left(L_1-L_2\right)\left(\theta_0+\varepsilon_n\right)-\alpha\ln\rho_n+2\pi i m_n \to A_0
\end{aligned}
\tag{3.23}
$$

Now, if $k_n\rho_n \to \infty$, then by the relation of A_k in (3.23) we deduce that $A_k = \infty$, a contradiction. If $k_n\rho_n \to a$, $a > 0$ then by the relation for A_{k-1} in (3.23), we get that $A_{k-1} = \infty$ (here we use $k \geq 2$), a contradiction. Hence we deduce that $k_n\rho_n \to 0$. Then from (3.23), we see that if $A_i \neq 0$ for some $2 \leq i \leq k$, then $A_{i-1} = \infty$. Thus $A_i = 0$ for $2 \leq i \leq k$.

We can assume that $A_1 \neq 0$ (since otherwise g is a constant function), and so $g(\zeta) = e^{A_0+A_1\zeta}$. By (3.23) and (3.2) we get

$$
\arg A_1 = \arg a_k + (k-1)\theta_0 = \frac{(k-1)(\pm\pi/2) + \arg a_k + (k-1)2\pi l}{k}
\tag{3.24}
$$

for some $0 \leq l \leq k-1$.

We observe that in (3.24) there are $2k$ different arguments.

By the fact that there must be some limit function g and by Lemma 3.1, we obtain that the possibility $z_n \to z_0 \neq 0$ gives (for every $-1 < \alpha < 1$) the collection

$$
\bigcup_{l=0}^{k-1}\left\{e^{A_0+A_1\zeta} : A_0 \in \mathbb{C},\ \arg A_1 = \frac{\arg a_k + (k-1)(\pm\pi/2) + (k-1)2\pi l}{k}\right\}
\tag{3.25}
$$

to $\Pi_\alpha(f)$.

We turn now to the case where $k_n z_n \to \infty$ and $z_n \to 0$.

Claim 3.3. $k_n \rho_n \to 0$.

Proof. If $k_n \rho_n \to \infty$, then in the case that $\frac{z_n}{\rho_n} \to \infty$, (3.21)-(3.23) hold and we get a contradiction by the relation for A_k in (3.23). If $\frac{z_n}{\rho_n} \to b \in \mathbb{C}$, we get a contradiction similarly to the parallel case in section 3.2 (see (3.15)).

If on the other hand, $k_n \rho_n \to a$, $0 < a < \infty$, then the relations in (3.23) hold, and by the relation of A_{k-1} we get that $A_{k-1} = \infty$, a contradiction and the claim is proven. $\square$

We can deduce now, as in the case where $z_n \to z_0 \neq 0$, that $g(\zeta) = e^{A_1 \zeta + A_0}$ and for A_1, A_0 the two last relations in (3.23) hold, respectively.

We separate now according to the value of α.

Case (B1) $\boldsymbol{0 < \alpha < 1}$.

We can assume that $\arg(z_n) \to \theta_0$.

Claim 3.4. *There is some* $0 \le l \le k-1$*, such that* $\pi/2 + 2\pi l \le \arg a_k + k\theta_0 \le 3\pi/2 + 2\pi l$.

Proof. If it is not the case, then we have $Re(P(k_n z_n)) \to +\infty$. Without loss of generality, we can assume that $\left\{ \frac{\ln \left| k_n z_n \right|}{\ln \rho_n} \right\}$ converges (in the extended sense). Now, if $\frac{\ln \left| k_n z_n \right|}{\ln \rho_n} \to b$, $0 < b \le \infty$, then $\frac{-\alpha \ln \rho_n}{Re[P(k_n z_n)]} \to 0$, and since $\frac{\ln \left| k_n z_n \right|}{Re[P(k_n z_n)]} \to 0$, we deduce that the real part of the left side of the relation for A_0 in (3.23) tends to $+\infty$, and this is a contradiction. If on the other hand $\frac{\ln \left| k_n z_n \right|}{\ln \rho_n} \to 0$, then since $\alpha > 0$ we derive the same conclusion and get a contradiction.

This completes the proof of the claim. $\square$

Hence we can write

$$\frac{2\pi l}{k}+\frac{\pi/2-\arg a_k}{k}\leq\theta_0\leq\frac{3\pi/2-\arg a_k}{k}+\frac{2\pi l}{k} \tag{3.26}$$

for some $0\leq l\leq k-1$.

We denote $\theta_1:=\arg A_1$, and by the relation for A_1 in (3.23) we have $\theta_1=\arg a_k+(k-1)\theta_0$ and thus

$$\arg a_k+\frac{k-1}{k}\left(\frac{\pi}{2}-\arg a_k+2\pi l\right)\leq\theta_1\leq\arg a_k+\frac{k-1}{k}\left(\frac{3\pi}{2}-\arg a_k+2\pi l\right)\ , \tag{3.27}$$

$0\leq l\leq k-1$.

We show now that for every θ_1 that satisfies (3.27), there is $g\in\Pi_\alpha(f)$, $g(\zeta)=e^{A_0+A_1\zeta}$ with $\arg A_1=\theta_1$.

Evidently it is enough for this purpose to show that for every θ_0 that satisfies (3.26), there are sequences $\{k_n\}$, $k_n\in\mathbb{N}$, $\{m_n\}$, $m_n\in\mathbb{Z}$ and $\{z_n\}$, $\{\rho_n\}$, $z_n\to 0$ with $\arg z_n\to\theta$, $\rho_n\to 0^+$, such that the relations (3.23) hold (with $0=A_2=\cdots=A_k$, $A_1\neq 0$ and $A_0\in\mathbb{C}$ arbitrary).

We first show it for θ_0 that satisfies (3.26) with sharp inequalities (and the corresponding θ_1 will satisfy (3.27) with sharp inequalities).

Indeed, for $n\geq 2$ define $k_n=n$, $\rho_n=\frac{1}{n^{1+\frac{k-1}{k}\frac{\ln\ln n}{\ln n}}}$ and $\hat{z}_n=e^{i\theta_0}(\frac{-\ln\rho_n}{n})^{\frac{1}{k}}$.

Observe that since $k\geq 2$, $k_n\rho_n\to 0$ and $k_n\hat{z}_n\to\infty$, we have

$$k_n\rho_n(k_n\hat{z}_n)^{k-1}=\frac{1}{n^{\frac{k-1}{k}\frac{\ln\ln n}{\ln n}}}\big[\big(1+\frac{k-1}{k}\frac{\ln\ln n}{\ln n}\big)\ln n\big]^{\frac{k-1}{k}}$$

$$=(1+\frac{k-1}{k}\frac{\ln\ln n}{\ln n})^{\frac{k-1}{k}}\underset{n\to\infty}{\to}1.$$

In addition we have $\frac{\left|k_n\hat{z}_n\right|^k}{-\ln\rho_n}=1$. By the choice of θ_0 (see (3.26)), we get

$$\frac{Re[P(n\hat{z_n})+(L_1-L_2)\ln|n\hat{z_n}|]}{|a_k(n\hat{z}_n)^k|}\underset{n\to\infty}{\to}\cos(\arg a_k+k\theta_0)<0\ ,$$

and then we get

(3.28)
$$\frac{-\alpha\ln\rho_n}{-Re[P(n\hat{z}_n)+(L_1-L_2)\ln|n\hat{z}_n|]}\to\frac{-\alpha}{|a_k|\cos(\arg a_k+k\theta_0)}>0.$$

Denote $C_0=\frac{-\alpha}{|a_k|\cos(\arg a_k+k\theta_0)}$. From (3.28) we deduce that for large enough n

$$\frac{C_0}{2}[-Re[P(n\hat{z}_n)+(L_1-L_2)\ln|n\hat{z}_n|]]<-\alpha\ln\rho_n$$
$$<2C_0[-Re(P(n\hat{z}_n))+(L_1-L_2)\ln|n\hat{z}_n|].$$

By the Mean Value Theorem there is some t_n, $\sqrt[k]{\frac{C_0}{2}}<t_n<\sqrt[k]{2C_0}$ such that

$$-Re[P(n\hat{z}_nt_n)+(L_1-L_2)\ln|n\hat{z}_nt_n|]=-\alpha\ln\rho_n. \tag{3.29}$$

(In fact, it is easy to see that every sequence $\{t_n\}$ of real numbers that satisfies (3.29) must satisfy $t_n\to\sqrt[k]{C_0}$.)

We set $z_n=t_n\hat{z}_n$ and then the relation for A_1 in (3.23) holds for some A_1 with $\arg A_1=\arg a_k+(k-1)\theta_0$. By (3.29) there are (after moving to subsequence if necessary, that will be denoted with no loss of generality with the same indices) integers m_n, $\boldsymbol{n\geq 2}$ such that the relation with regard to A_0 in (3.23) holds for some $A_0\in\mathbb{C}$.

Morever, since $k_n\rho_n\to 0$ and $k_nz_n\to\infty$, we deduce that the relations for $A_2,\dots,A_k$ in (3.23) hold and give $0=A_2=A_3=\cdots=A_k$.

The fulfillment of these relations in (3.23) means that

$$\frac{f(k_nz_n+k_n\rho_n\zeta)}{\rho_n^\alpha}\overset{\chi}{\Rightarrow}e^{A_0+A_1\zeta}\ .$$

By (2) of Lemma 3.1, every function g, $g(\zeta)=e^{a\zeta+b}$ with $\arg a=\arg a_k+(k-1)\theta_0=\theta_1$, and arbitrary $b\in\mathbb{C}$ is in $\Pi_\alpha(f)$.

Now suppose that θ_1 is equal to the left or to the right side of (3.27). Without loss of generality,

$$\theta_1=\arg a_k+\frac{k-1}{k}\left[\frac{3\pi}{2}-\arg a_k+2\pi l\right]\ ,\ 0\leq l\leq k-1\ .$$

Then we take an increasing sequence, $\left\{\theta_1^{(l)}\right\}_{l=1}^{\infty}$ such that

$$\arg a_k + \frac{k-1}{k}\left(\frac{\pi}{2} - \arg a_k + 2\pi l\right) < \theta_1^{(l)} \underset{l\to\infty}{\nearrow} \theta_1.$$

By the case of sharp inequality in (3.27), for every $l \geq 1$, correspond sequences $z_n^{(l)} \underset{n\to\infty}{\to} 0$, $\rho_n^{(l)} \underset{n\to\infty}{\to} 0^+$ such that

$$\frac{f(z_n^{(l)} + n\rho_n^{(l)}\zeta)}{\rho_n^{(l)\alpha}} \underset{n\to\infty}{\overset{\chi}{\Rightarrow}} e^{e^{i\theta_1^{(l)}}\zeta} .$$

Since

$$e^{e^{i\theta_1(l)}\zeta} \underset{l\to\infty}{\Rightarrow} e^{e^{i\theta_1}\zeta} ,$$

then in a similiar way to the case $k_n z_n \to \infty$, $k_n\rho_n \to a$ in Case (B) of section 3.2, we deduce the existence of sequences $\rho_n \to 0^+$, $z_n \to 0$, and $\{k_n\}$ such that

$$\frac{f(k_n z_n + k_n\rho_n\zeta)}{\rho_n^{\alpha}} \underset{n\to\infty}{\Rightarrow} e^{e^{i\theta_1}\zeta}.$$

As usual, by Lemma 3.1 every $g(\zeta) = e^{a\zeta+b}$ with $\arg a = \theta_1$ and arbitrary $b \in \mathbb{C}$ belongs to $\Pi_\alpha(f)$.

In order to determine explicitly $\Pi_\alpha(f)$, we need to find the range of θ_1 in (3.27). For $k = 2$ we have

$$(3.30) \qquad l = 0: \quad \frac{\pi}{4} + \frac{\arg a_2}{2} \leq \theta_1 \leq \frac{3\pi}{4} + \frac{\arg a_2}{2}$$

$$l = 1: \quad \frac{5\pi}{4} + \frac{\arg a_2}{2} \leq \theta_1 \leq \frac{7\pi}{4} + \frac{\arg a_2}{2}.$$

There are two distinct intervals with sum of length π.

Claim 3.5. *For $k \geq 3$ the range of θ_1 in (3.27) is* $[0, 2\pi]$.

Proof. Denote for $0 \leq l \leq k-1$, the general interval in (3.27) by $I_l = [\varepsilon_l, \delta_l]$. The length of I_l is $|I_l| = \pi\frac{k-1}{k}$ and $\varepsilon_{l+1} - 2\pi + \frac{2\pi}{k} = \varepsilon_l$. Thus it is enough to show that $\frac{k-1}{k}\pi \geq \frac{2\pi}{k}$ and $\frac{k-1}{k}\pi + (k-1)\frac{2\pi}{k} \geq 2\pi$. It is easy to see that these two inequalities are satisfied for $k \geq 3$. The claim is proven. □

As a result, from the claim and from Lemma 3.1, we get that for $k \geq 3$ the posiibility $z_n \to 0$, $k_n z_n \to \infty$ gives the collection (for $0 < \alpha < 1$)

$$\left\{e^{a\zeta+b} : A \neq 0,\ b \in \mathbb{C}\right\} \tag{3.31}$$

to $\Pi_\alpha(f)$.

We turn now to the complementary case.

Case (B2) $\boldsymbol{k_n z_n \to \infty}$, $\boldsymbol{z_n \to 0}$, $\boldsymbol{-1 < \alpha < 0}$

Here again, as $f_{n,\alpha}(\zeta) \overset{\chi}{\Rightarrow} g(\zeta)$ if and only if $(\frac{1}{f})_{n,-\alpha}(\zeta) \overset{\chi}{\Rightarrow} (\frac{1}{g})(\zeta)$ and since $\frac{1}{f} = \frac{1}{R}e^{-P}$, i.e., a function of the same type we get the following.

For $k = 2$, observe that $\frac{1}{e^{a\zeta+b}} = e^{-a\zeta-b}$ and $\arg(-a) = \pi + \arg a$ and also the leading coefficient of $-P(z)$ has the argument $\arg(-a_2) = \pi + \arg(a_2)$. So we substitute in (3.27) (or in (3.30)) these values (or $\arg(a) - \pi$ and $\arg(a_2) - \pi$ resp.) instead of θ_1 and $\arg(a_2)$, respectively, to get

$$\frac{\arg a_2}{2} + \frac{3\pi}{4} \leq \theta_1 \leq \frac{5\pi}{4} + \frac{\arg a_2}{2} \quad \text{or} \quad \frac{7\pi}{4} + \frac{\arg a_2}{2} \leq \theta_1 \leq \frac{9\pi}{4} + \frac{\arg a_2}{2}. \tag{3.32}$$

Observe that the set of values of $a \in \mathbb{C}$ correesponds to (3.32) is the complement (up to the boundary) of the set of values of $a \in \mathbb{C}$ corresponding to (3.30).

For $k \geq 3$ we get the collection

$$\left\{e^{a\zeta+b} : a \neq 0, b \in \mathbb{C}\right\} \tag{3.33}$$

to $\Pi_\alpha(f)$, exactly as in (3.31).

The last case to treat is

$\boldsymbol{k_n z_n \to \infty}$, $\boldsymbol{z_n \to 0}$, $\boldsymbol{\alpha = 0}$.

In this case as we saw also, $k_n \rho_n \to 0$. Also the relations in (3.23) hold and $A_i = 0$ for $2 \leq i \leq k$, and $A_1 \neq 0$.

We can assume, without loss of generality, that $\arg(z_n) \to \theta_0$. From the relations for A_0 in (3.23), we get

$$\arg a_k + k\theta_0 = \pm\frac{\pi}{2} + 2\pi l \quad \text{for some } l \in \mathbb{Z}\ . \tag{3.34}$$

And by the relation for A_1 in (3.23), we get

(3.35)
$$\theta_1 := \arg A_1 = \arg a_k + (k-1)\theta_0 = \frac{(k-1)(\pm\frac{\pi}{2}) + \arg a_k + (k-1)2\pi l}{k},$$

$0 \le l \le k-1$.

In the other direction we show now that every function of the form $g(\zeta) = e^{a\zeta+b}$, with $\theta_1 = \arg(a)$, that satisfies (3.35) is obtained in $\Pi_0(f)$.

Indeed, set $\theta_1 = \theta_1(\theta_0) = \arg a_k + (k-1)\theta_0$.

For every θ_0 that satisfies (3.34) and for every $m \ge 1$, there exist according to (3.25) sequences $z_n^{(m)} \underset{n\to\infty}{\to} \frac{1}{m}e^{i\theta_0}$, $\rho_n^{(m)} \underset{n\to\infty}{\to} 0^+$ and $\left\{k_n^{(m)}\right\}_{n=1}^{\infty}$ such that

$$f(k_n^{(m)}z_n^{(m)} + k_n^{(m)}\rho_n^{(m)}\zeta) \overset{\chi}{\underset{n\to\infty}{\Rightarrow}} e^{e^{i\theta_0}\zeta}\ .$$

Hence, we get as in the case $k_n z_n \to \infty$ in Case (B) in Section 3.2 that $g(\zeta) = e^{e^{i\theta_0}\zeta}$ is attained as a limit function with $z_n \to 0$ (and $\arg z_n = \theta_0$) and $\rho_n \to 0^+$. Then as usual by Lemma 3.1, we obtain that every $g(\zeta) = e^{a\zeta+b}$, with $\arg a = \theta_1$ where θ_1 is as in (3.35). Thus this option gives the collection

(3.36)
$$\bigcup_{l=0}^{k-1}\left\{e^{a\zeta+b} : b \in \mathbb{C}, \arg a = ((k-1)(\pm)\frac{\pi}{2} + \arg a_k + (k-1)2\pi l)/k\right\}$$

to $\Pi_0(f)$.

Observe that not as in the cases $0 < \alpha < 1$, $-1 < \alpha < 0$, this case does not add to $\Pi_\alpha(f)$, (here $\alpha = 0$), new functions.

Now we can finally collect all the limit functions to fix $\Pi_\alpha(f)$ for $-1 < \alpha < 1$ in the case $k \ge 2$.

$\boldsymbol{\alpha = 0}$

For every $k \geq 2$ we get by (3.18), (3.25) (and (3.36))

$$\Pi_0(f) = \{f(a\zeta + b) : a > 0, b \in \mathbb{C}\} \bigcup$$

$$\left\{e^{a\zeta+b} : b \in \mathbb{C}, \arg a = \frac{\arg a_k + (k-1)(\pm\frac{\pi}{2}) + (k-1)2\pi l}{k}, \quad 0 \leq l \leq k-1\right\} .$$

$\boldsymbol{0 < \alpha < 1}$

For $k = 2$ we get by (3.19) **and** (3.25) and (3.30)

$$\Pi_\alpha(f) = \left\{\bigcup_{i=1}^{m} \left\{e^{P(\gamma_i)}\tilde{R}_{\gamma_i}(\gamma_i)(a\zeta + b)^{l_i} : a > 0, \quad b \in \mathbb{C}\right\}\right\} \bigcup$$

$$\left\{e^{a\zeta+b} : \ b \in \mathbb{C}, \ \frac{\pi}{4} + \frac{\arg a_2}{2} \leq \arg a \leq \frac{3\pi}{4} + \frac{\arg a_2}{2} \quad \text{or}\right.$$

$$\left.\frac{5\pi}{4} + \frac{\arg a_2}{2} \leq \arg a \leq \frac{7\pi}{4} + \frac{\arg a_2}{2}\right\} .$$

For $k \geq 3$ we get by (3.19), (3.25) **and** (3.31)

$$\Pi_\alpha(f) = \Big[\bigcup_{i=1}^{m} \left\{e^{P(\gamma_i)}\tilde{R}_{\gamma_i}(\gamma_i)(a\zeta + b)^{l_i} : a > 0, b \in \mathbb{C}\right\}\Big] \bigcup$$

$$\left\{e^{a\zeta+b} : a \neq 0, b \in \mathbb{C}\right\} .$$

$\boldsymbol{-1 < \alpha < 0}$

For $k = 2$ we get by (3.20), (3.25) **and** (3.32)

$$\Pi_\alpha(f) = \Big[\bigcup_{i=1}^{l} \left\{e^{P(\beta_i)}\hat{R}_{\beta_i}(\beta_i)(a\zeta + b)^{-j_i} : a > 0, \ b \in \mathbb{C}\right\}\Big] \bigcup$$

$$\left\{e^{a\zeta+b} : b \in \mathbb{C}, \frac{-\pi}{4} + \frac{\arg a_2}{2} \leq \arg a \leq \frac{\pi}{4} + \frac{\arg a_2}{2} \quad \text{or}\right.$$

$$\left.\frac{3\pi}{4} + \frac{\arg a_2}{2} \leq \arg a \leq \frac{5\pi}{4} + \frac{\arg a_2}{2}\right\} .$$

For $k \geq 3$ (3.20), (3.25) **and** (3.33) give

$$\Pi_\alpha(f) = \Big[\bigcup_{i=1}^{l}\Big\{e^{P(\beta_i)}\hat{R}_{\beta_i}(\beta_i)(a\zeta+b)^{-j_i} : a>0, b\in\mathbb{C}\Big\}\Big]\bigcup$$
$$\Big\{e^{a\zeta+b} : a\neq 0, b\in\mathbb{C}\Big\}.$$

The proof of Theorem 1 is completed.

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